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Decomposition of two functions in the orthogonality equation

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Abstract. The aim of this paper is to solve the orthogonality equation with two unknown functions. This problem was posed by J. Chmieliński during two international conferences.

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1. Introduction

Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\langle \cdot | \cdot \rangle$ denote the inner product and $\|\cdot\|$ the norm associated with it. We shall not distinguish between the symbols used for $\mathcal{H}$ and $\mathcal{K}$. The Banach space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathcal{L}(\mathcal{H}; \mathcal{K})$. It is known that $h \colon \mathcal{H} \to \mathcal{K}$ is a solution of the orthogonality equation:

$$\forall_{x,y\in\mathcal{H}} \quad \langle h(x) | h(y) \rangle = \langle x | y \rangle$$

if and only if h is a linear isometry (or equivalently $h \in \mathcal{L}(\mathcal{H}; \mathcal{K})$ and $h^*h = I_{\mathcal{H}}$).

The following considerations have been inspired by the talks of J. Chmieliński during the 15th ICFEI and CUTS (see [3, p. 95] and [4, p. 145]). Namely, we will solve the generalized orthogonality equation:

$$\forall_{x,y\in\mathcal{H}} \quad \langle f(x) | g(y) \rangle = \langle x | y \rangle, \tag{1}$$

with two unknown functions $f, g \colon \mathcal{H} \to \mathcal{K}$. The paper [1] also deals with Eq. (1) and similar topics. We need the following lemma for further investigations.

Lemma 1. *Let $f, g \colon \mathcal{H} \to \mathcal{K}$ satisfy Eq.* (1) *and $\overline{\mathrm{Lin} f(\mathcal{H})} = \mathcal{K}$. Then g is linear.*

Proof. Fix $y \in \mathrm{Lin} f(\mathcal{H})$. Then $y = \sum_{k=1}^n \alpha_k f(x_k)$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$, $x_1, \ldots, x_n \in \mathcal{H}$. Thus we have

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$$\begin{aligned}\langle y|g(u+\beta w)\rangle &= \left\langle \sum_{k=1}^{n}\alpha_k f(x_k)|g(u+\beta w)\right\rangle = \sum_{k=1}^{n}\alpha_k \langle f(x_k)|g(u+\beta w)\rangle\\ &= \sum_{k=1}^{n}\alpha_k \langle x_k|u+\beta w\rangle = \sum_{k=1}^{n}\alpha_k \left(\langle x_k|u\rangle + \overline{\beta}\langle x_k|w\rangle\right)\\ &= \sum_{k=1}^{n}\alpha_k \left(\langle f(x_k)|g(u)\rangle + \overline{\beta}\langle f(x_k)|g(w)\rangle\right)\\ &= \sum_{k=1}^{n}\langle \alpha_k f(x_k)|g(u)+\beta g(w)\rangle\\ &= \left\langle \sum_{k=1}^{n}\alpha_k f(x_k)|g(u)+\beta g(w)\right\rangle\\ &= \langle y|g(u)+\beta g(w)\rangle, \ u,w\in\mathcal{H}, \quad \beta\in\mathbb{K}.\end{aligned}$$

The set $\mathrm{Lin} f(\mathcal{H})$ is dense in $\mathcal{K}$, whence $g(u+\beta w) = g(u)+\beta g(w)$ for $u, v \in \mathcal{H}$, $\beta \in \mathbb{K}$. This means that g is linear. $\square$

Some well-known facts will be useful for further considerations. Let $T \in \mathcal{L}(\mathcal{K};\mathcal{H})$. Then there exists a unique $T^* \in \mathcal{L}(\mathcal{H};\mathcal{K})$ such that

$$\forall_{x\in\mathcal{K}}\ \forall_{y\in\mathcal{H}}\ \ \langle Tx|y\rangle_{\mathcal{H}} = \langle x|T^*y\rangle_{\mathcal{K}}.$$

Moreover,

$$\text{if}\quad \forall_{x\in\mathcal{H}}\ \vartheta\|x\| \leqslant \|Tx\|, \quad \text{then } T(\mathcal{H}) \text{ is closed.} \tag{2}$$

It is also known that

$$T \text{ is invertible if and only if } T^* \text{ is invertible.} \tag{3}$$

Lemma 2. *Let $T_1, T_2\colon \mathcal{H} \to \mathcal{K}$ be linear maps and satisfy*

$$\forall_{x,y\in\mathcal{H}} \quad \langle T_1(x)|T_2(y)\rangle = \langle x|y\rangle\,. \tag{4}$$

Assume that $T_1(\mathcal{H})$ is dense. Then T_2 is continuous, i.e., $T_2 \in \mathcal{L}(\mathcal{H};\mathcal{K})$.

Proof. Fix a sequence $(x_n)_{n=1,2,\ldots}$ such that $x_n \in \mathcal{H}$ and $x_n \to x_o$ for some $x_o \in \mathcal{H}$. Suppose that $T_2x_n \to z$ for some $z \in \mathcal{H}$. It suffices to show that $T_2x_o = z$ and apply The Closed Graph Theorem. Fix $y \in \mathcal{H}$ and notice that $\langle x_n - x_o|y\rangle \to 0$. On the other hand $\langle x_n - x_o|y\rangle \stackrel{(4)}{=} \langle T_2x_n - T_2x_o|T_1y\rangle \to \langle z - T_2x_o|T_1y\rangle$. Thus we get

$$\forall_{y\in\mathcal{H}} \quad \langle z - T_2x_o|T_1y\rangle = 0.$$

Since $T_1(\mathcal{H})$ is dense in $\mathcal{K}$, it follows that $z - T_2x_o = 0$. We have shown that the graph of T_2 is closed. $\square$

Lemma 3. *Let $T, S \in \mathcal{L}(\mathcal{H};\mathcal{K})$ satisfy the equation*

$$\forall_{x,y\in\mathcal{H}} \quad \langle T(x)|S(y)\rangle = \langle x|y\rangle\,. \tag{5}$$

Then T is invertible if and only if S is invertible.

Proof. Assume that T is invertible. Then $\langle x|y\rangle = \langle Tx|Sy\rangle = \langle x|T^*Sy\rangle$ for all $x, y \in \mathcal{H}$. Thus we have $I_{\mathcal{H}} = T^*S$. It follows from Lemma (3) that T^* is invertible. Therefore $(T^*)^{-1} = S$, hence S is invertible. The proof of the reverse is the same. □

2. Main result

In this section, we solve functional Eq. (1). Now we can state and prove the main result of the paper.

Theorem 4. *Let $f, g\colon \mathcal{H} \to \mathcal{K}$ satisfy Eq. (1) if and only if there exist suitable closed subspaces $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \subset \mathcal{K}$ such that $\mathcal{K} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$, $\mathcal{M}_k \perp \mathcal{M}_j$ (for $k \neq j$) and f, g can be written as the following decomposition*

$$f = A + \varphi, \quad g = (A^*)^{-1} + \gamma,$$

for some invertible $A \in \mathcal{L}(\mathcal{H}; \mathcal{M}_1)$ and for some mappings $\varphi\colon \mathcal{H} \to \mathcal{M}_2$, $\gamma\colon \mathcal{H} \to \mathcal{M}_3$.

Proof. The implication "$\Leftarrow$" is immediate. We start with proving "$\Rightarrow$". It is clear that $\mathcal{K} = \overline{\mathrm{Lin} f(\mathcal{H})} \oplus \overline{\mathrm{Lin} f(\mathcal{H})}^{\perp}$, whence there are two mappings $T_1 : \mathcal{H} \to \overline{\mathrm{Lin} f(\mathcal{H})}$ and $\varphi_1 : \mathcal{H} \to \overline{\mathrm{Lin} f(\mathcal{H})}^{\perp}$ such that $g(x) = T_1(x) + \varphi_1(x)$ for all $x \in \mathcal{H}$. It is easy to check that f, T_1 satisfy

$$\forall_{x,y\in\mathcal{H}} \quad \langle f(x)|T_1(y)\rangle = \langle x|y\rangle\,. \tag{6}$$

Indeed,

$$\begin{aligned}\langle x|y\rangle = \langle f(x)|g(y)\rangle = \langle f(x)|T_1(y) + \varphi_1(y)\rangle &= \langle f(x)|T_1(y)\rangle + \langle f(x)|\varphi_1(y)\rangle \\ &= \langle f(x)|T_1(y)\rangle + 0 = \langle f(x)|T_1(y)\rangle\,.\end{aligned}$$

We have shown that f, T_1 satisfy (6). Moreover, the set $\mathrm{Lin} f(\mathcal{H})$ is dense in $\overline{\mathrm{Lin} f(\mathcal{H})}$. It follows from 1 that T_1 is linear.

There exists the closed subspace $\mathcal{M} \subset \overline{\mathrm{Lin} f(\mathcal{H})}$ such that $\overline{T_1(\mathcal{H})} \perp \mathcal{M}$ and $\overline{\mathrm{Lin} f(\mathcal{H})} = \overline{T_1(\mathcal{H})} \oplus \mathcal{M}$. Therefore there are two mappings $T_2 : \mathcal{H} \to \overline{T_1(\mathcal{H})}$ and $\varphi_2 : \mathcal{H} \to \mathcal{M}$ such that $f(x) = T_2(x) + \varphi_2(x)$ for all $x \in \mathcal{H}$. In a similar way we prove that T_2, T_1 satisfy

$$\forall_{x,y\in\mathcal{H}} \quad \langle T_2(x)|T_1(y)\rangle = \langle x|y\rangle\,. \tag{7}$$

Now, we can consider the linear operators $T_2 : \mathcal{H} \to \overline{T_1(\mathcal{H})}$ and $T_1 : \mathcal{H} \to \overline{T_1(\mathcal{H})}$ instead of $T_1 : \mathcal{H} \to \overline{\mathrm{Lin} f(\mathcal{H})}$. By Lemma 1 and (7), T_2 is linear. By Lemma 2 and (7), T_2 is continuous, i.e., $T_2 \in \mathcal{L}(\mathcal{H}; \overline{T_1(\mathcal{H})})$.

There is the closed subspace $\mathcal{N} \subset \overline{T_1(\mathcal{H})}$ such that $\overline{T_1(\mathcal{H})} = \overline{T_2(\mathcal{H})} \oplus \mathcal{N}$ and $T_2(\mathcal{H}) \perp \mathcal{N}$. Hence there are two mappings $T_3 : \mathcal{H} \to \overline{T_2(\mathcal{H})}$ and $\varphi_3 : \mathcal{H} \to \mathcal{N}$ such that $T_1(x)=T_3(x)+\varphi_3(x)$ for all $x\in\mathcal{H}$. In a similar way we prove that T_3 is linear and T_2, T_3 satisfy

$$\forall_{x,y\in\mathcal{H}} \quad \langle T_2(x)|T_3(y)\rangle = \langle x|y\rangle\,. \tag{8}$$

Now, we consider the linear operator $T_2 \in \mathcal{L}(\mathcal{H};\overline{T_2(\mathcal{H})})$ (instead of $T_2\in\mathcal{L}(\mathcal{H};\overline{T_1(\mathcal{H})})$). Let us consider also the linear operator $T_3\colon\mathcal{H}\to\overline{T_2(\mathcal{H})}$. Applying again Lemma 2 and (8) we can say that T_3 is continuous, i.e., $T_3\in\mathcal{L}(\mathcal{H};\overline{T_2(\mathcal{H})})$. Now, we get

$$\|x\|^2=\langle x|x\rangle \overset{(8)}{=} \langle T_2x|T_3x\rangle \leqslant \|T_2x\|\cdot\|T_3x\| \leqslant \|T_2x\|\cdot\|T_3\|\cdot\|x\|$$

for all $x\in\mathcal{H}$. It follows from the above inequality that

$$\forall_{y\in\mathcal{H}} \quad \frac{1}{\|T_3\|}\cdot\|x\| \leqslant \|T_2x\|. \tag{9}$$

Thus $T_2(\mathcal{H})$ is closed (see 2). Thus we obtain that $T_2(\mathcal{H})=\overline{T_2(\mathcal{H})}$ and T_2, $T_3\in\mathcal{L}(\mathcal{H};T_2(\mathcal{H}))$. It follows from (9) that T_2 in injective, hence T_2 is invertible. Therefore $T_3\in\mathcal{L}(\mathcal{H};T_2(\mathcal{H}))$ is also invertible by Theorem 3 and (8).

Finally, we define $\mathcal{M}_1 := T_2(\mathcal{H})$, $\mathcal{M}_2 := \mathcal{M}$ and $\mathcal{M}_3 := \mathcal{N}\oplus\overline{\operatorname{Lin} f(\mathcal{H})}^{\perp}$. Next, we define $\varphi\colon\mathcal{H}\to\mathcal{M}_2$, $\gamma\colon\mathcal{H}\to\mathcal{M}_3$ by $\varphi:=\varphi_2$, $\gamma:=\varphi_3+\varphi_1$. Moreover, we define $A\in\mathcal{L}(\mathcal{H};\mathcal{M}_1)$ by $A:=T_2$. Clearly A is invertable and $(A^*)^{-1}=T_3$. Thus we get $f=A+\varphi$ and $g=(A^*)^{-1}+\gamma$ and $\mathcal{K}=\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3$, where $\mathcal{M}_k\perp\mathcal{M}_j$ for $k\neq j$. $\square$

Corollary 5. *Suppose that* $\dim\mathcal{H}<\infty$. *Let* $f,g\colon\mathcal{H}\to\mathcal{H}$ *satisfy Eq.* (1). *Then* $f,g\in\mathcal{L}(\mathcal{H};\mathcal{H})$ *and* $f=(g^*)^{-1}$.

Proof. By the above theorem we have a decomposition $f=A+\varphi$, $g=(A^*)^{-1}+\gamma$ for some invertible $A\in\mathcal{L}(\mathcal{H};\mathcal{M}_1)$ and some mappings $\varphi\colon\mathcal{H}\to\mathcal{M}_2$, $\gamma\colon\mathcal{H}\to\mathcal{M}_3$ (where $\mathcal{H}=\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3$ and $\mathcal{M}_k\perp\mathcal{M}_j$). Since $A\in\mathcal{L}(\mathcal{H};\mathcal{M}_1)$ is invertible and $\dim\mathcal{H}<\infty$, it follows that $\mathcal{M}_1=\mathcal{H}$ and $\mathcal{M}_2=\{0\}=\mathcal{M}_3$. So $\varphi=0=\gamma$. $\square$

Corollary 6. *Let* $f\colon\mathcal{H}\to\mathcal{K}$ *satisfy the equation*

$$\forall_{x,y\in\mathcal{H}} \quad \langle f(x)|f(y)\rangle = \langle x|y\rangle\,. \tag{10}$$

Then f *is a linear isometry.*

Proof. By Theorem 4 we have a decomposition $f=A+\varphi$, $f=(A^*)^{-1}+\gamma$ for some $A\in\mathcal{L}(\mathcal{H};\mathcal{M}_1)$ and $\varphi\colon\mathcal{H}\to\mathcal{M}_2$, $\gamma\colon\mathcal{H}\to\mathcal{M}_3$. Thus we get $A+\varphi=(A^*)^{-1}+\gamma$. Since $\mathcal{M}_2\perp\mathcal{M}_3$, we get $\varphi=0=\gamma$. Hence $f=A$. Then f is a linear mapping and by (10), f is a linear isometry. $\square$

Similar investigations have been carried out in the manuscript [2]. Namely, Eq. (1) and its stability, as well as the approximate orthogonality preserving property are considered in [2].

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